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Application of Modified Three Parameter Weibull Distributions to Brittle Material Design

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Summary

The three parameter Weibull distribution is used as a basis for probabilistic design with brittle materials. Beam fracture data are analysed by a new technique to obtain the characteristic Weibull parameters which are then used in conjunction with the theory of independent action to predict failure in a multiaxial stress state.

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CONTENTS

1 INTRODUCTION

Most materials behave in a brittle manner particularly if operating temperatures are low and rates of loading are high. However, there are many materials which are brittle even at ambient temperatures and slow rates of loading. Typical examples are; most engineering ceramics, tungsten below the ductile-brittle transition temperature, graphites, carbons, concrete, cement and compressed powder compounds. The characterisation of these materials, the manufacture of components and their response to load require special treatment.

The measurement of the strength of a material is usually made for one of two purposes; either to compare the relative merits of two materials or to provide design data for structural components. If the material is ductile the stress concentrations which may arise in the test pieces are redistributed through local plastic flow. A brittle material is far less forgiving in its response to load and if a flaw becomes critical it will usually propagate and result in catastrophic failure. Mechanical testing of brittle materials requires accurate specimen alignment and its neglect can cause premature failure leading to spurious test results. This is particularly true for the simple tensile test and even though it would simplify the numerical analysis it is rarely chosen for brittle material characterisation. More suitable test configurations in which alignment is not so critical are the flexure of beams and the annular bending or diametric compression of discs.

For brittle materials it is common to observe a large scatter in the fracture data of nominally identical test pieces. This is due to the fact that volume and surface defects are randomly distributed throughout the material and vary in occurrence, orientation and severity. Compared with that of most other structural materials, the variability of strength of brittle materials is high. This is reflected in the analysis of strength test data where large coefficients of variation are observed - typically 15%-20%, and values as high as 30% are not uncommon. These figures imply that a deterministic strength and an associated factor of safety are no longer sensible design parameters. Instead a statistical approach is required to account for the material variability.

The distribution function generally adopted in brittle material design is the one due to Weibull. This distribution is derived using a weak link hypothesis and its use allows the component failure probability to be expressed in terms of the failure characteristics of an elemental volume or surface. The Weibull equation can also model fracture data with skewed distributions and if the three parameter equation is adopted an allowance can be made for a non-zero threshold strength, ie a strength below which the probability of failure is zero.

The weak link hypothesis leads to a size effect relationship in which large components are predicted to be weaker than smaller ones since it is more likely that a critical flaw will be encountered as the volume or surface area increases. For materials with low variability (ie most metals), the size effect is negligible and strength data are not quoted with respect to component size. However, for brittle materials where the variability is significant, strength must be related to either a unit volume or a unit area depending on whether volume or surface defects are considered significant.

The characterisation of brittle materials requires careful experimentation and is normally achieved by analysing the fracture data obtained from flexural tests on slender beams. The same geometry can be used to study both volume and surface defects providing the appropriate analysis is applied in

each case. Once the characteristic failure properties of a material have been evaluated, the failure probability of a component subject to a multiaxial stress state can be examined. The development of probabilistic failure theories which allow this have been the subject of considerable attention during recent years.

In section 2 of this report the three parameter Weibull equation is introduced and its applications to brittle material design are described. Section 3 presents a new method for the evaluation of the Weibull parameters based upon a non-linear least squares analysis. The generalisation of the simple Weibull equation for non-uniform stress fields and its subsequent application to the analysis of volume and surface defects are given in Section 4. Section 5 gives details of a multiaxial stress theory for predicting the failure probability of certain types of brittle materials.

In another report by the authors (Ref 1), various experimental results are analysed and the statistical analysis developed in this work is assessed.

2 WEIBULL DISTRIBUTION

From a given set of fracture data the frequency distribution of strength can be constructed by recording the number of failures at the various stress levels. For many materials the frequency distribution is strongly skewed and the theoretical distribution chosen for any subsequent analysis must be capable of describing this. It is also observed that there is a threshold strength below which no component will fail and this implies the existence of a lower positive bound, below which the probability of failure is zero. These two features immediately rule out the normal distribution which is symmetric and is defined over a range extending from minus to plus infinity (Ref 2). Using a normal distribution would result in a finite probability of failure value for a component under zero load.

A distribution which has been used extensively to characterise brittle materials is the one due to Weibull (Ref 3). The frequency distribution for the three parameter equation in terms of an applied stress σ is

$$
\begin{aligned}
F &= \frac{m}{\sigma_o}\left(\frac{\sigma - \sigma_T}{\sigma_o}\right)^{m-1} \exp\left[-\left(\frac{\sigma - \sigma_T}{\sigma_o}\right)^{m}\right] & \sigma>\sigma_T, \\
F &= 0 & \sigma<\sigma_T,
\end{aligned}
\qquad (1)
$$

and the corresponding cumulative probability distribution is given by the equation

$$
\begin{aligned}
P &= 1 - \exp\left[-\left(\frac{\sigma - \sigma_T}{\sigma_o}\right)^{m}\right] & \sigma>\sigma_T, \\
P &= 0 & \sigma<\sigma_T.
\end{aligned}
\qquad (2)
$$

The Weibull parameters m, σ_o and σ_T are obtained by fitting the above distribution to the experimental data. In Equation 2 the threshold stress σ_T defines the stress below which the probability of failure is zero. Frequency and cumulative distribution curves with $m = 3$ and $\sigma_o = 1$ for σ_T = 0.0, 0.5, 1.0 are shown in Figure 1. Changing the threshold value while keeping the other two parameters fixed results in a simple translation of the distribution curves. The physical meaning of the Weibull modulus m can be seen from Figure 2 where the frequency and cumulative probability

distributions are plotted for $\sigma_T = 0$, $\sigma_0 = 1$ and $m = 2, 3, 4$. As m increases the spread of the frequency distribution decreases, becoming a single spike as m tends to ∞. This corresponds to a step change in the cumulative distribution. Symmetric distributions, eg the normal distribution, are said to have zero skewness and this occurs in the Weibull distribution when $m = 3.602$. It is also interesting to note that all the probability curves in Figure 2 pass through the same point given by the coordinates

$$\sigma = \sigma_T + \sigma_o \,, \qquad P = 1 - 1/e \,. \tag{3}$$

The normalising parameter σ_o affects the 'scale' of the distribution. This is clearly illustrated in Figure 3 where frequency and cumulative distribution curves are shown with $m = 7$, $\sigma_T = 0$ for $\sigma_o = 200, 300, 400, 500, 600$.

Once the Weibull parameters have been evaluated from a given set of fracture data, the mean value and coefficient of variation can be calculated from the respective formulae (Ref 4)

$$\bar{\sigma} = \sigma_T + \sigma_o \, \Gamma \, (1 + 1/m) \,, \tag{4}$$

$$C = \frac{\sqrt{\Gamma \, (1 + 2/m) - (\Gamma \, (1 + 1/m))^2}}{\Gamma \, (1 + 1/m) + \sigma_T/\sigma_o} \,. \tag{5}$$

Here $\Gamma(z)$ is the gamma function of z. Simplifications occur when $\sigma_T = 0$ and Equation 1 reduces to the two parameter distribution. The analysis of fracture data based on this assumption usually yields m values which are greater than 5 and this results in the approximate formulae

$$\bar{\sigma} = \sigma_o \left[1 - \frac{0.57}{m} \right] , \tag{6}$$

$$C = \frac{1.27}{m + 0.55} \tag{7}$$

From the last equation it is seen that for a two parameter Weibull distribution the Weibull modulus m is an inverse measure of material variability.

3 EVALUATION OF THE WEIBULL PARAMETERS

The evaluation of the Weibull parameters can be achieved by curve fitting, the method of moments, or the maximum likelihood technique. Curve fitting includes linear plotting and non-linear least squares estimation (Refs 2,5). For the two parameter distribution there is little to choose between the above methods and reliable results can be achieved in each case. For the three parameter distribution the choice of method is important and ill conditioning can arise for some numerical procedures. In this section a new approach is presented that is based on a non-linear least squares analysis which appears to offer improved consistency over previously documented least squares techniques.

In Equation 1 the Weibull parameters refer to population values. For most engineering applications only small samples of data will be available, so the parameters calculated will be estimates of the true parameters. As the sample size increases, these estimates will approach the population values. Confidence limits associated with the small sample estimates are clearly important and this topic is discussed elsewhere by the authors (Ref 1).

3.1 Plotting Position

If the fracture data are arranged in ascending order so that

$$\sigma = \sigma_i \ , \ \sigma_{i+1} > \sigma_i \ , \ i = 1,2.....n \ , \qquad (8)$$

an estimate of the probability value associated with each σ_i may be calculated. The true rank P_i is unknown but is distributed for the ith observation according to the density function (Ref 6)

$$g\ (P_i) = \frac{n\ !}{(i-1)\ !\ (n-i)\ !}\ (P_i)^{i-1}\ (1 - P_i)^{n-i}\ . \qquad (9)$$

In practice, the mean or expected value $\bar{P}_i$ of the rank P_i is often used where

$$\bar{P}_i = \int_0^1 P_i\ g\ (P_i)\ d\ P_i \qquad (10)$$

which simplifies to

$$\bar{P}_i = \frac{i}{n+1}\ . \qquad (11)$$

However, computations have shown that this assumption leads to biased estimates of the Weibull parameters (Ref 7) resulting from inaccurate representations of the fracture data in the lower probability region. An alternative approach is to choose the median of the distribution. These are calculated by solving the integral equation

$$\int_0^{\tilde{P}_i} g\ (P_i)\ d\ P_i = 0.5 \qquad (12)$$

to find a value of the median rank, $\tilde{P}_i$ so that the associated cumulative distribution function is 0.5. For n less than 20 the $\tilde{P}_i$ value is obtained by solving Equation 12 numerically. Results for sample sizes of n ranging from 1 to 20 are tabulated in Reference 6 which also gives the approximate formula

$$\tilde{P}_i = \frac{i - (1 - \ln 2) - (2\ln 2 - 1)\left[\frac{i-1}{n-1}\right]}{n} \qquad (13)$$

for use when $n > 20$. Adopting these values for P_i leads to a better representation of the data.

3.2 Calculation of the Weibull Parameters

Once the median ranks are known the Weibull parameters may be calculated. Many techniques have been presented for evaluating these quantities and some of the methods are outlined in a paper by Schneider and Palazotto (Ref 5). Most methods fall into two categories; a trial and error graphical approach in which the data are linearised or a least squares minimisation based on the probability curve. The former method normally uses a minimisation scheme based on the differences

$$\delta_i = \ln\ln\left[1/(1 - P(\sigma_i))\right] - \ln\ln\left[1/(1 - P_i)\right] . \tag{14}$$

In the latter method use is made of the difference scheme

$$\delta_i = P(\sigma_i) - P_i \tag{15}$$

where $P(\sigma_i)$ is the value of P calculated from Equation 2 using σ_i, and P_i is the selected plotting position. Both schemes rely on a minimisation with respect to probability values and computations have shown that significant changes in the calculated parameters may be caused by small changes in the data. This is particularly noticeable for the threshold stress which in some cases can have a calculated value outside the valid range.

$$0 < \sigma_T < \sigma_1 . \tag{16}$$

More reliable calculations can be made for the Weibull parameters if a least squares analysis is adopted which is based on the difference scheme

$$\delta_i = \sigma(P_i) - \sigma_i = \hat{\sigma}_i - \sigma_i . \tag{17}$$

In equation 17 $\hat{\sigma}_i$ satisfies the transformed Weibull relation

$$\hat{\sigma}_i = \sigma_T + \sigma_o \left[\ln\left(1/(1 - P_i)\right)\right]^{1/m} \tag{18}$$

for the selected plotting position P_i. Minimising the sum of the square of the differences

$$\Delta = \sum_{i=1}^{n} (\delta_i)^2 \tag{19}$$

with respect to each Weibull parameter leads to the least squares equations

$$n\,\sigma_T + \sigma_o \sum_{i=1}^{n} W_i^{1/m} = \sum_{i=1}^{n} \sigma_i , \tag{20}$$

$$\sigma_T \sum_{i=1}^{n} W_i^{1/m} + \sigma_o \sum_{i=1}^{n} W_i^{2/m} = \sum_{i=1}^{n} \sigma_i W_i^{1/m} , \tag{21}$$

$$\sigma_T \sum_{i=1}^{n} W_i^{1/m} \ln W_i + \sigma_o \sum_{i=1}^{n} W_i^{2/m} \ln W_i = \sum_{i=1}^{n} \sigma_i W_i^{1/m} \ln W_i , \tag{22}$$

where

$$W_i = \ln\left(1/(1 - P_i)\right) . \tag{23}$$

Equations 20 and 21 may be solved simultaneously to express σ_o and σ_T in terms of the Weibull modulus m, ie

$$\sigma_o = \frac{n \sum_{i=1}^{n} \sigma_i W_i^{1/m} - \sum_{i=1}^{n} W_i^{1/m} \sum_{i=1}^{n} \sigma_i}{n \sum_{i=1}^{n} W_i^{2/m} - \sum_{i=1}^{n} W_i^{1/m} \sum_{i=1}^{n} W_i^{1/m}} , \tag{24}$$

$$\sigma_T = \frac{\sum_{i=1}^{n} \sigma_i \sum_{i=1}^{n} W_i^{2/m} - \sum_{i=1}^{n} W_i^{1/m} \sum_{i=1}^{n} \sigma_i W_i^{1/m}}{n \sum_{i=1}^{n} W_i^{2/m} - \sum_{i=1}^{n} W_i^{1/m} \sum_{i=1}^{n} W_i^{1/m}} . \tag{25}$$

Substituting for σ_o and σ_T into Equation 22 leads to a transcendental equation for m. Although its final form is too complex to be written down in this text the resulting equation may be readily solved numerically on a high speed computer. In the root solving routine an initial value of $m = 1.0001$ can be chosen to start the iterative solution procedure. For values of $m<1$ the Weibull equation degenerates into a physically unrealistic form for the studies under investigation.

The above analysis has been developed on the assumption that the threshold value σ_T is unknown. If the threshold value can be prescribed certain simplifications occur. For analytical reasons it is convenient to introduce the difference notation

$$\delta_i = \ln (\sigma (P_i) - \sigma_T) - \ln (\sigma_i - \sigma_T) \tag{26}$$

which via Equation 17 reduces to

$$\delta_i = \ln (\hat{\sigma}_i - \sigma_T) - \ln (\sigma_i - \sigma_T) . \tag{27}$$

Performing a least squares analysis based on Equation 27 gives the explicit expressions for the Weibull modulus and normalising factor

$$m = \frac{n \sum_{i=1}^{n} W_i^2 - \sum_{i=1}^{n} W_i \sum_{i=1}^{n} W_i}{n \sum_{i=1}^{n} W_i \ln (\sigma_i - \sigma_T) - \sum_{i=1}^{n} W_i \sum_{i=1}^{n} \ln (\sigma_i - \sigma_T)} ,$$

$$\sigma_o = \exp \left[\frac{\sum_{i=1}^{n} W_i^2 \sum_{i=1}^{n} \ln (\sigma_i - \sigma_T) - \sum_{i=1}^{n} W_i \sum_{i=1}^{n} W_i \ln (\sigma_i - \sigma_T)}{n \sum_{i=1}^{n} W_i^2 - \sum_{i=1}^{n} W_i \sum_{i=1}^{n} W_i} \right] , \tag{28}$$

where

$$W_i = \ln (\ln (1/(1 - P_i))) . \tag{29}$$

4 BRITTLE MATERIAL CHARACTERISATION

The Weibull distribution defined by Equation 2 is applicable to the analysis of tensile test fracture data. However, because of the practical differences previously mentioned, flexural tests on slender beams are normally used to characterise brittle materials.

When a slender beam is subjected to bending, the stress is uniaxial but non-uniform. Equation 2 is no longer applicable in its present form and requires modification to account for the non-uniform stress field. If the beam is divided into N elements, each assumed to be subject to a uniform stress, the failure probability of the assembly is

$$P = 1 - \prod_{i=1}^{N} (1 - P_i^{(e)}) \tag{30}$$

where $\prod_{i=1}^{N} (1 - P_i^{(e)})$ is the product of the individual reliabilities of the N elements (Ref 8). Applying this argument to an elementary volume ΔV, and allowing ΔV to tend to zero leads in the limit to the following expression:

$$\begin{aligned} P^{(V)} &= 1 - \exp\left[-\frac{1}{V}\int_V \left(\frac{\sigma - \sigma_T}{\sigma_o}\right)^m dV\right] & \sigma > \sigma_T\ , \\ P^{(V)} &= 0 & \sigma < \sigma_T\ . \end{aligned} \tag{31}$$

A similar result can be derived for area defects

$$\begin{aligned} P^{(A)} &= 1 - \exp\left[-\frac{1}{A}\int_A \left(\frac{\sigma - \sigma_T}{\sigma_o}\right)^m dA\right] & \sigma > \sigma_T\ , \\ P^{(A)} &= 0 & \sigma < \sigma_T\ . \end{aligned} \tag{32}$$

Here V and A respectively denote the volume and surface area of the beam over which integration is performed. For simplicity the symbols m, σ_T and σ_o have been used in both Equations 31 and 32, but the parameter values will differ depending on the particular type of defect being investigated.

If a slender beam is subjected to a constant bending moment M, the bending stress is given by

$$\sigma^{(b)} = \frac{My}{I} \tag{33}$$

where I is the 'moment of inertia' of the beam cross section about the neutral axis, and y is the distance of a point in the cross section from that axis. A convenient way of producing the above stress state is to subject the beam to four point loading shown diagrammatically in Figure 4. For the purpose of this analysis failure loads are only recorded when fracture occurs in the central span where the bending moment is constant.

Considerable simplifications can be made to the theory if rectangular beams are selected for testing. In this case the volume and surface integrals appearing in Equations 31 and 32 can be evaluated in closed form to yield (Ref 9)

$$\begin{aligned} P_i^{(V)} &= 1 - \exp\left[-\frac{1}{2(m+1)}\left(1 - \frac{\sigma_T}{\sigma_i^{(b)}}\right)\left(\frac{\sigma_i^{(b)} - \sigma_T}{\sigma_o}\right)^m\right] & \sigma_i^{(b)} > \sigma_T\ , \\ P_i^{(V)} &= 0 & \sigma_i^{(b)} < \sigma_T\ , \end{aligned} \tag{34}$$

$$P_i(A) = 1 - \exp\left[-\frac{1}{2(h+b)}\left(\frac{\sigma_i(b) - \sigma_T}{\sigma_o}\right)^m\left\{\frac{h}{m+1}\left(1 - \frac{\sigma_T}{\sigma_i(b)}\right) + b\right\}\right] \qquad \sigma_i(b) > \sigma_T,$$

(35)

$$P_i(A) = 0 \qquad \sigma_i(b) < \sigma_T.$$

In the above equations b and h respectively denote the breadth and height of the beam and $\sigma_i(b)$ is the ith maximum bending stress

$$\sigma_i(b) = \frac{6M_i}{b\,h^2}\ . \tag{36}$$

For analytical reasons it is convenient in Equation 34 to define the auxiliary variables

$$\begin{aligned} \overset{*}{m} &= m + 1 \quad , \\ \overset{*}{\sigma}_o &= (2\ (m+1)\ \sigma_o{}^m)^{\frac{1}{m+1}} \quad , \\ \overset{*}{\sigma}_T &= \sigma_T \quad . \end{aligned} \tag{37}$$

Equation 34 can then be rewritten as

$$P(V) = 1 - \exp\left[-\frac{1}{\sigma_i(b)}\left(\frac{\sigma_i(b) - \overset{*}{\sigma}_T}{\overset{*}{\sigma}_o}\right)^{\overset{*}{m}}\right] \qquad \sigma_i(b) > \overset{*}{\sigma}_T\ ,$$

(38)

$$P(V) = 0 \qquad \sigma_i(b) < \overset{*}{\sigma}_T\ ,$$

and rearranged, for a particular bend stress $\sigma_i^{(b)}$, to give

$$\sigma_i(b) = \overset{*}{\sigma}_T + \overset{*}{\sigma}_o\left[\sigma_i(b)\ \ln\ (1/(1 - P_i(V)))\right]^{1/\overset{*}{m}} \tag{39}$$

When Equation 39 is compared with Equation 18 it is seen that they are identical in form provided that the term W_i defined by Equation 23 is replaced by

$$W_i = \sigma_i(b)\ \ln\ (1/(1 - P_i(V))) \tag{40}$$

It therefore follows that the solution for the auxiliary parameter set $\overset{*}{m}$,

$\overset{*}{\sigma}_o$, $\overset{*}{\sigma}_T$ may be obtained from Equations 20 to 25 with W_i defined by Equation 40 and σ_i replaced by $\sigma_i^{(b)}$. The actual Weibull parameters defining the uniaxial strength distribution for volume defects are then given by

$$
\begin{aligned}
m &= \overset{*}{m} - 1 \quad , \\
\sigma_o &= \left(\frac{\overset{*}{\sigma}_o \, \overset{*}{m}}{2\overset{*}{m}}\right)^{1/(\overset{*}{m} - 1)} \quad , \\
\sigma_T &= \overset{*}{\sigma}_T \quad .
\end{aligned}
\tag{41}
$$

To obtain the Weibull parameters from a four point bend test assuming failure due to surface defects, Equation 35 has to be solved for m, σ_o and σ_T. Again for analytical reasons it is convenient to introduce the auxiliary variables

$$
\begin{aligned}
\overset{*}{\sigma}_o &= \left(\frac{2\ (h + b)}{b}\right)^{1/m} \sigma_o \quad , \\
\overset{*}{m} &= m \quad , \\
\overset{*}{\sigma}_T &= \sigma_T \quad , \\
\varepsilon_i &= \frac{h}{b\ (\overset{*}{m} + 1)} \left(1 - \frac{\overset{*}{\sigma}_T}{\sigma_i^{(b)}}\right) \quad .
\end{aligned}
\tag{42}
$$

With these changes of variables, Equation 35 can be expressed in the form

$$
\begin{aligned}
P_i^{(A)} &= 1 - \exp\left[-\left(\frac{\sigma_i^{(b)} - \overset{*}{\sigma}_T}{\overset{*}{\sigma}_o}\right)^{\overset{*}{m}} (1 + \varepsilon_i)\right] \qquad \sigma_i^{(b)} > \overset{*}{\sigma}_T \ , \\
P_i^{(A)} &= 0 \qquad \sigma_i^{(b)} < \overset{*}{\sigma}_T \ ,
\end{aligned}
\tag{43}
$$

and rearranged for a particular $\sigma_i^{(b)}$ to give

$$
\sigma_i^{(b)} = \overset{*}{\sigma}_T + \overset{*}{\sigma}_o \left[\frac{1}{1 + \varepsilon_i} \ln\ (1/(1 - P_i^{(A)}))\right]^{1/\overset{*}{m}} \ . \tag{44}
$$

Since the quantity ε_i is a function of $\overset{*}{m}$ and $\overset{*}{\sigma}_T$ it is not possible to solve Equation 44 directly for the Weibull parameters. However, a solution can be obtained iteratively if restrictions are imposed on the cross sectional dimensions of the beam. By selecting beams that have a h/b ratio of less

than unity and noting that for all cases $\overset{*}{m}$ is greater than unity, the term ε_i will always be less than 0.5. Under these circumstances it is possible to arrive at values for the Weibull parameters by making use of the iterative equations

$$\sigma_i^{(b)} = \overset{*}{\sigma}_T^{(k+1)} + \overset{*}{\sigma}_o^{(k+1)} \left[\frac{1}{1 + \varepsilon_i^{(k+1)}} \ln\left(1/1(1 - P_i^{(A)})\right) \right]^{\frac{1}{\overset{*}{m}(k+1)}} \quad k = 1,2\ldots$$

$$\varepsilon_i^{(k+1)} = \frac{h}{b\,(\overset{*}{m}^{(k)} + 1)} \left(1 - \frac{\overset{*}{\sigma}_T^{(k)}}{\sigma_i^{(b)}}\right) \quad . \tag{45}$$

Comparison of Equation 45 with Equation 18 shows that they are of the same form. Hence the auxiliary parameter set may be obtained using Equation 45 with Equations 20 to 25 when W_i in those equations is replaced by

$$W_i = \frac{1}{1 + \varepsilon_i^{(k+1)}} \ln\left(1/(1 - P_i^{(A)})\right) \ . \tag{46}$$

The initial values $\overset{*}{m}(1), \overset{*}{\sigma}_o(1), \overset{*}{\sigma}_T(1)$ required to start the iteration are obtained by solving Equation 43 with $\varepsilon_i^{(1)}$ set equal to zero. The iterative process is terminated when the relative difference between each parameter is less than some prescribed tolerance. Computations have shown that this is usually achieved in less than ten iterations for a prescribed tolerance of 1.0×10^{-5}. The parameters of the uniaxial distribution are then obtained using the final values of $\overset{*}{m}^{(k)}$, $\overset{*}{\sigma}_o^{(k)}$ and $\overset{*}{\sigma}_T^{(k)}$ in Equation 42.

4.1 Size Effect Relation

The average tensile failure strength $\bar{\sigma}$ associated with volume or surface flaws depends on the volume or surface area of the component under test. This phenomenon is known as the 'size effect'. For the flexural tests described previously, the volume and surface area under consideration corresponds to the region between the two inner knife edges. It can be shown that the 'size effect' is governed by the relation (Ref 8)

$$\frac{\bar{\sigma}_{V_1} - \sigma_T}{\bar{\sigma}_{V_2} - \sigma_T} = \left(\frac{V_2}{V_1}\right)^{1/m} \tag{47}$$

where $\bar{\sigma}_{V_1}$ and $\bar{\sigma}_{V_2}$ are the respective strengths of volumes V_1 and V_2. Using this relationship a unit volume average tensile failure strength $\bar{\sigma}_v$ can be introduced and related to any other volume strength σ_V by the equation

$$\frac{\bar{\sigma}_v - \sigma_T}{\bar{\sigma}_V - \sigma_T} = \left(\frac{V}{v}\right)^{1/m} . \qquad (48)$$

Although v has the numerical value of unity it is retained so that the above equation has the correct dimensional form. Eliminating σ_o from Equation 2 in favour of $\bar{\sigma}_v$ using Equations 4 and 48 results in the modified Weibull equation

$$\begin{aligned} P &= 1 - \exp\left[- \Gamma\{(1 + 1/m)\}^m \frac{V}{v}\left(\frac{\sigma - \sigma_T}{\bar{\sigma}_v - \sigma_T}\right)^m\right] \qquad \sigma>\sigma_T , \\ P &= 0 \qquad \sigma<\sigma_T . \end{aligned} \qquad (49)$$

In this expression the failure probability of a component of volume V which is subjected to a uniform uniaxial stress σ, is given in terms of the characteristic quantities m, σ_T and $\bar{\sigma}_v$. A similar relationship may be derived for surface area defects.

5 MULTIAXIAL FAILURE THEORY

When a structure is loaded, the state of stress at a point will in general be multiaxial and can be uniquely defined in terms of the three principal stresses σ_1, σ_2, σ_3. For a given set of loading conditions these stresses vary in magnitude and direction as a function of position within the structure. By treating the loaded component as an assembly of elements, it can be readily shown by considering the reliability of each element that the extension of Equation 49 to a non-uniform uniaxial stress field is

$$\begin{aligned} P &= 1 - \exp\left[- \{\Gamma(1 + 1/m)\}^m \frac{V}{v} \int_V \left(\frac{\sigma - \sigma_T}{\bar{\sigma}_v - \sigma_T}\right)^m \frac{dV}{V}\right] \qquad \sigma>\sigma_T , \\ P &= 0 \qquad \sigma<\sigma_T . \end{aligned} \qquad (50)$$

It is not possible to extend the above equation rigorously to include a non-uniform multiaxial stress state; instead it is necessary to postulate a criterion of failure which will describe this condition. Many different theories have been presented for various materials (Refs 10, 11), and it is the author's view that it is unlikely that a single theory will emerge that will describe the behaviour of all materials. Individual theories will need to be developed and assessed for the particular materials under investigation. On the basis of this philosophy, a theory is presented here which gives reasonable results for a range of graphites used in rocket nozzle applications in the UK.

As a working model it is assumed that the reliability of an elemental volume is calculated by considering the effect of each of the principal stresses independently. This is the criterion of independent action, originally proposed by Stanley at al (Ref 8). With this assumption the reliability of each elementary volume due to each principal stress in turn, and hence that of the entire structure, can be assessed. A generalisation of Equation 50, which is based on this assumption and is applicable to anisotropic materials subject to both compressive and tensile stresses, is given by

$$P = 1 - \exp\left[-\left\{\Gamma(1 + 1/m)\right\}^m \frac{V}{\bar{V}} \cdot S_v\right] \qquad \sigma_i > \sigma_T^{(T)}\ ,\ \sigma_i > \sigma_T^{(C)} \tag{51}$$

$$P = 0 \qquad \sigma_i < \sigma_T^{(T)}\ ,\ \sigma_i > \sigma_T^{(C)}$$

where

$$S_v = \sum_{i=1}^{3} \int_V \left(\frac{\sigma_i - \sigma_{T_i}(\sigma_i)}{\bar{\sigma}_{v_i}(\sigma_i) - \sigma_{T_i}(\sigma_i)}\right)^m \frac{dV}{\bar{V}}\ ,$$

$$\sigma_{T_i}(\sigma_i) = \sigma_{T_i}{}^{(T)}\ ,\ \bar{\sigma}_{v_i}(\sigma_i) = \bar{\sigma}_{v_i}{}^{(T)} \text{ for } \sigma_i > \sigma_{T_i}{}^{(T)} \tag{52}$$

$$\sigma_{T_i}(\sigma_i) = \sigma_{T_i}{}^{(C)}\ ,\ \bar{\sigma}_{v_i}(\sigma_i) = \bar{\sigma}_{v_i}{}^{(C)} \text{ for } \sigma_i > \sigma_{T_i}{}^{(C)}\ .$$

In Equation 51 the summation i=1,2,3 refers to the principal directions. The quantities $\sigma_{T_i}{}^{(T)}$ and $\sigma_{T_i}{}^{(C)}$, i=1,2,3 are the threshold strengths in the three principal directions for tension and compression respectively. Similarly, the unit volume uniaxial strengths in the principal directions for tension and compression are $\sigma_{v_i}{}^{(T)}$ and $\sigma_{v_i}{}^{(C)}$, i=1,2,3. The compressive stresses and strengths in these equations are negative quantities.

To utilise Equation 51 the values of m, $\bar{\sigma}_v$ and σ_T need to be known in the direction of the principal stresses at all points throughout the material volume. Determination of these quantities for all possible orientations would require extensive mechanical testing. It is therefore important from a practical point of view to introduce some simplifying assumptions in order to reduce the number of strength tests needed. For a particular grade of graphite it has been shown (Ref 12) (by a Weibull analysis of flexure data) that the Weibull modulus is independent of orientation. It has also been found that the value of the unit volume average strength $\bar{\sigma}_v$ for an orientation defined by the direction cosines (l_1, l_2, l_3) is given by

$$\sigma_v = \left[\left(\frac{l_1}{\bar{\sigma}_{v_1}}\right)^2 + \left(\frac{l_2}{\bar{\sigma}_{v_2}}\right)^2 + \left(\frac{l_3}{\bar{\sigma}_{v_3}}\right)^2\right]^{-1/2} \tag{53}$$

where $\bar{\sigma}_{v_i}$, i=1,2,3 are the respective values of $\bar{\sigma}_v$ in the principal material directions. It is postulated that when a three parameter Weibull distribution is utilised (with σ_T>0), Equation 53 remains valid. Past experience has shown that the average strength for a set of data is approximately constant whether it is calculated from the results of a two or three parameter analysis. It may be possible to assume that for the three parameter case the Weibull modulus is independent of orientation, and that σ_T is described by a strength ellipsoid similar to that of $\bar{\sigma}_v$. The variation of σ_T is only an hypothesis and its exact variation may be difficult to define since its value is determined effectively by extrapolation which can be very sensitive to small changes in the data.

Equations 50 to 53 have been developed assuming volume defects. The corresponding expressions for surface defects are obtained by reformulating the analysis in terms of the surface integrals.

6 CONCLUSION

This report describes a new technique for evaluating the parameters of a Weibull distribution from a set of uniaxial fracture data. An extension to this technique allows characterisation of a brittle material from flexural tests on beams in which the stress field is uniaxial but non-uniform. Using this method, the characteristic Weibull parameters for either surface or volume defects can be obtained. In order that multiaxial stress states can be analysed, the theory of independent action is then invoked. This failure theory makes use of either surface or volume defect Weibull parameters in the failure probability calculation. The method of characterisation and failure prediction using the theory of independent action are verified experimentally in another report by the authors (ref 1). In Reference 1, experimental results are analysed and methods are developed which enable the probabilistic design technique to be applied.

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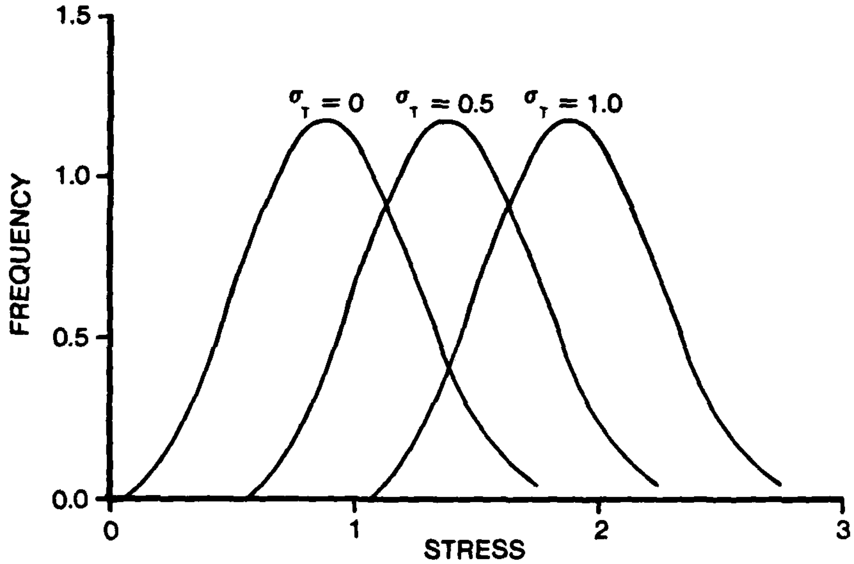

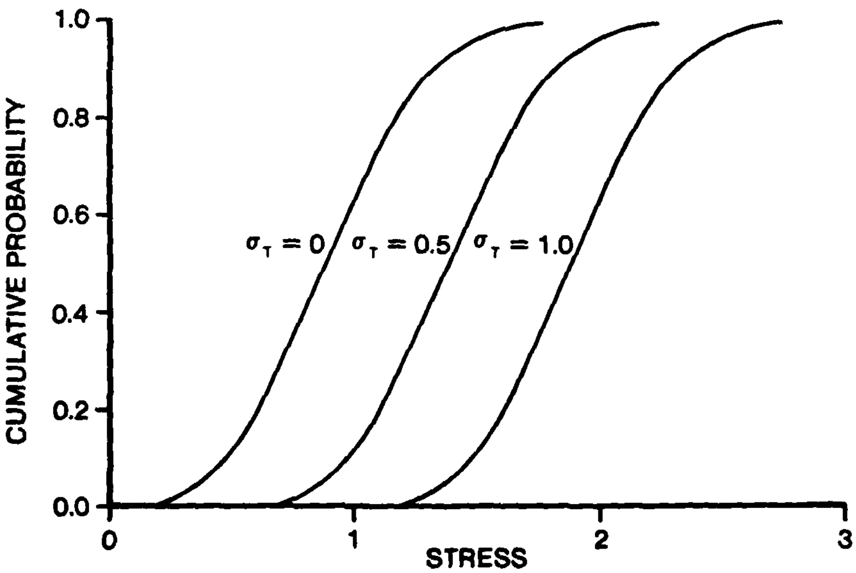

FIG. 1 WEIBULL DISTRIBUTION FUNCTIONS AND CUMULATIVE PROBABILITY CURVES FOR $m = 3$, $\sigma_0 = 1$ AND VARYING σ_T VALUES

FIG. 2

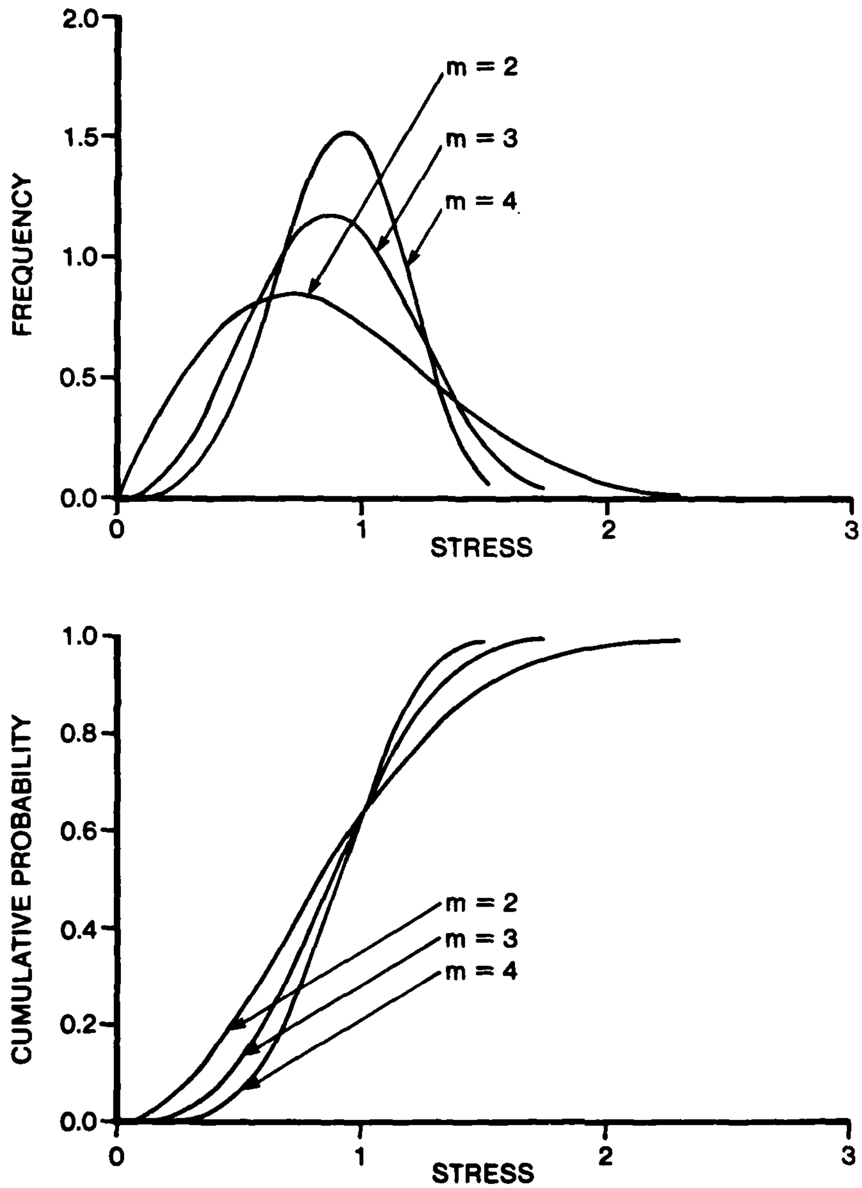

FIG. 2 WEIBULL DISTRIBUTION FUNCTIONS AND CUMULATIVE PROBABILITY CURVES FOR $\sigma_T = 0$, $\sigma_0 = 1$ AND VARYING m VALUES

FIG. 3

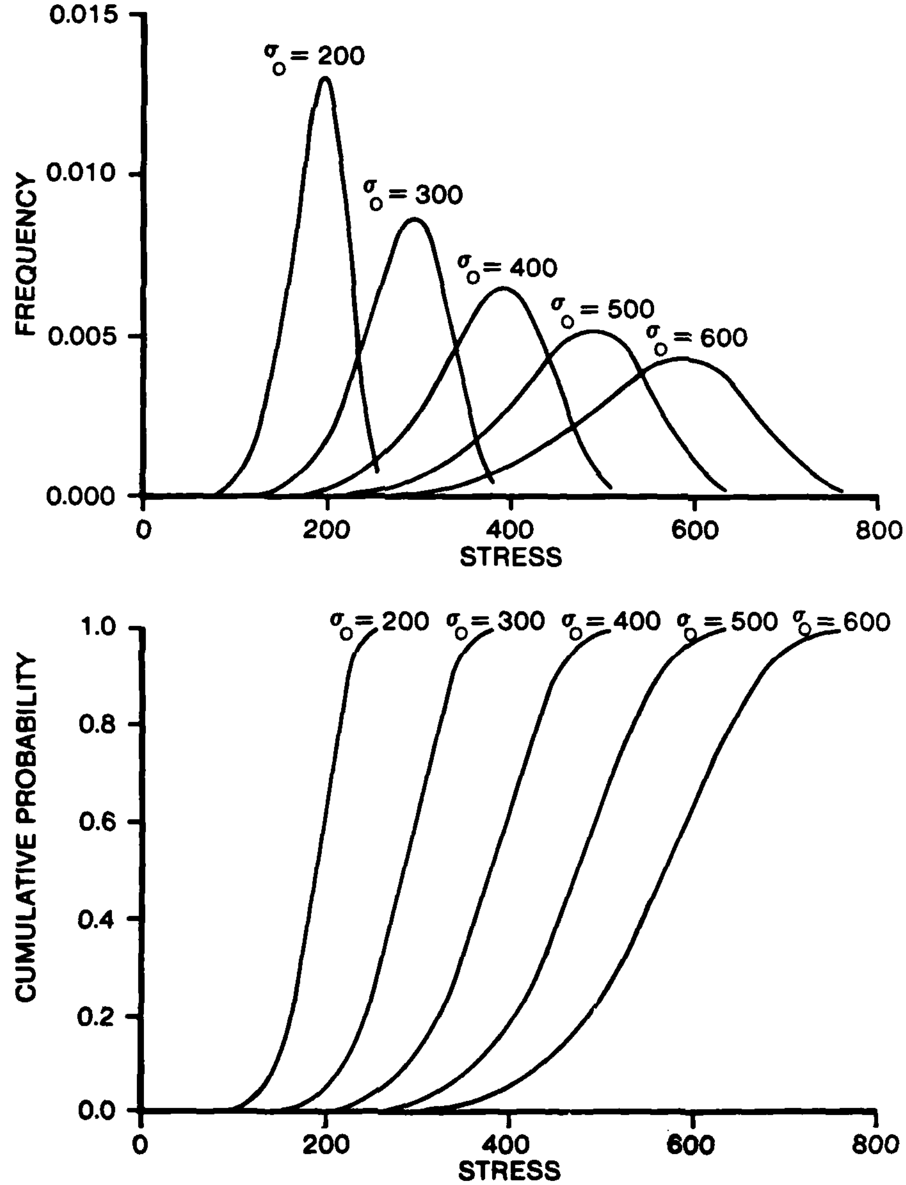

FIG. 3 WEIBULL DISTRIBUTION FUNCTIONS AND CUMULATIVE PROBABILITY CURVES FOR $m = 7$, $\sigma_T = 0$ AND VARYING σ_o VALUES

FIG. 4

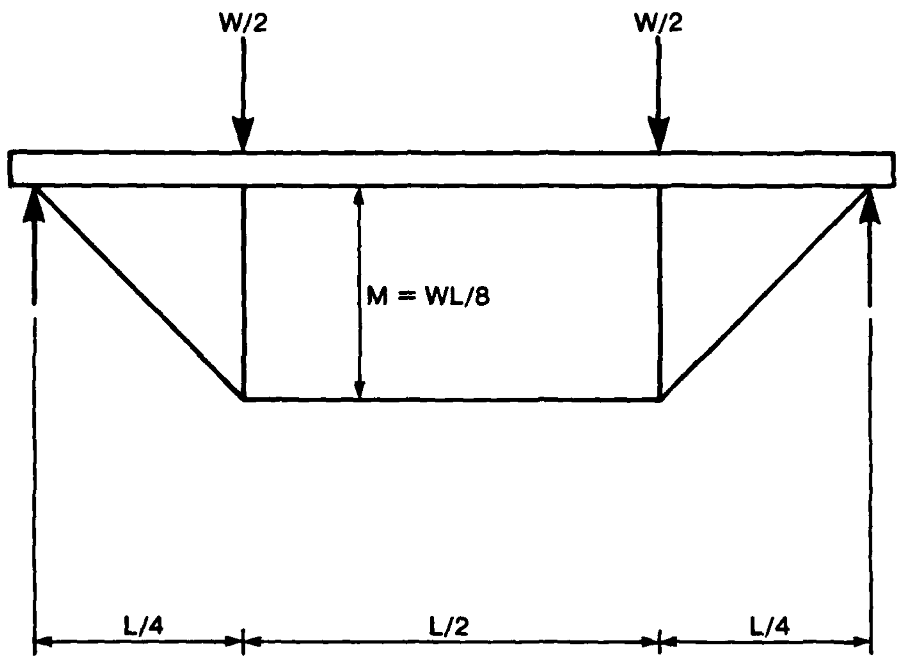

FIG. 4 DIAGRAM SHOWING FOUR POINT FLEXURE LOADING AND THE RESULTING BENDING MOMENT DIAGRAM

REPORT DOCUMENTATION PAGE

(Notes on completion overleaf)

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Abstract The three parameter Weibull distribution is used as a basis for probabilistic design with brittle materials. Beam fracture data are analysed by a new technique to obtain the characteristic Weibull parameters which are then used in conjunction with the theory of independent action to predict failure in a multiaxial stress state.